Assignment 4

The Rule RAA. In this form of argument, the conclusion X is established by showing that the assumption of ~X leads to a contradiction. For example, in the logic puzzle in the first assignment, the speaker B says “Both of us are Knaves.” We can use this statement to show that B cannot be a Knight. For assume that he is a Knight. Then what he says must be true so both A and B are Knaves and so B is a Knave. This contradicts our assumption, so we now know, using *reduction ad absurdum* reasoning, that B cannot be a Knight. Therefore, he must be a Knave.

To introduce this into our proof theory, we need to define a contradiction. Two sentences are contradictory precisely when one is the exact negation of the other. For example, all of the following pairs of sentences are contradictions:
P, ~P   Q, ~Q   PvQ, ~(PvQ)   ~P&Q, ~(~P&Q)   Q→ ~R, ~(Q→ ~R). However, the following are not contradictions: PvR, ~Pv~R   P→Q, ~P→Q   P→Q, P→ ~Q. In none of these examples is one sentence exactly the negation of the other.

The RAA rule is then stated as follows: if from an assumption Z we can derive a contradiction X and ~X, then we can infer the denial of Z. The line ~Z will depend on all of the assumptions that X depends on plus the assumptions that ~X depends on minus the assumption Z. Schematically, the rule RAA might be represented by:

k	(k)	Z	A
k,a,b	(m)	X	
k, b, c	(n)	~X	
a,b,c	(o)	~Z	m,n RAA (k)

Note that RAA has a similar restriction to →I in that the line k which you wish to deny must be an assumption.

RAA Strategy:

Before using RAA, check first to see if other strategies can be applied, such as →I. RAA typically leads to more complicated proofs, so exhaust other options first. However, RAA can be particularly helpful when your goal is a disjunction. Once you have assumed the denial of your goal, any contradiction will serve to allow RAA. When deciding what to aim for, it is usually easiest to contradict some sentence already occurring in the proof. See Example 2.

EXAMPLE 1   P→(Q→R), P→(~Q→R) ├ P→R

Step 1. Use all of our standard strategies first ; P→R is a conditional; so assume its antecedent and try to prove its consequent. Then →E

1	(1)	P→(Q→R)	A
2	(2)	P→(~Q→R)	A
3	(3)	P	A

simplifies our problem a bit.

1,3	(4) Q→R	1,3 →E
2,3	(5) ~Q→R	2,3 →E
1,2,3	(n-1) R	
1,2	(n) P→R	→I

Step 2. At this point, there is no obvious way to proceed so RAA seems like a good strategy to follow. Since the new goal is R, I will assume its denial, ~R, and then attempt to prove a contradiction.

1	(1) P→(Q→R)	A
2	(2) P→(~Q→R)	A
3	(3) P	A
1,3	(4) Q→R	1,3 →E
2,3	(5) ~Q→R	2,3 →E
6	(6) ~R	A
	X	
	~X	
1,2,3	(n-1) R	RAA
1,2	(n) P→R	→I

Step 3. In this case, it is clear what contradiction to prove. By MT I can infer Q and also ~Q.

1	(1) P→(Q→R)	A
2	(2) P→(~Q→R)	A
3	(3) P	A
1,3	(4) Q→R	1,3 →E
2,3	(5) ~Q→R	2,3 →E
6	(6) ~R	A
1,3,6	(7) ~Q	4,6 MT
2,3,6	(8) Q	5,6 MT
1,2,3	(9) R	7,8 RAA (6)
1,2	(10) P→R	9 →I (3)

EXAMPLE 2 ~(~Pv~Q) $\vdash$ P&Q

Step 1. This is an instance of DeMorgan's Laws, so we could prove it in one step if we were allowed that rule. But it can also be proved using only the primitive rules. In this case, we are trying to prove a conjunction, so I will prove each part and then use &I.

1	(1) ~(~Pv~Q)	A
	? P	new goal
	? Q	new goal
1	(n) P&Q	&I

Step 2. There is no obvious way to proceed, so I will try to use RAA. To prove P, assume ~P and then derive a contradiction. Then use RAA to infer P. Then repeat this same procedure to prove Q.

1	(1) ~(~Pv~Q)	A
2	(2) ~P	A
	X	
	~X	

	(?) P	RAA
	~Q	
	Q	RAA
1	(n) P&Q	&I

Step 3. Which contradiction should I try to derive? It is often easiest to simply contradict something already in the proof. Since ~(~Pv~Q) appears on line 1 of the proof, we could try to prove ~Pv~Q and then we would have a contradiction (besides, how else are we going to use line 1?) Once we set our goal, it is easy to see how to get it through vI.

1	(1) ~(~Pv~Q)	A
2	(2) ~P	A
2	(3) ~Pv~Q	2 vI
1	(4) P	1,3 RAA (2)
5	(5) ~Q	A
5	(6) ~Pv~Q	5 vI
1	(7) Q	1,6 RAA (5)
1	(8) P&Q	4,7 &I

In this problem, it is extremely important that we derive ~Pv~Q from ~Q in the second half of our problem rather than simply using lines 1 and 3 again. If we did that, our conclusion Q would still depend on line 2.

EXAMPLE 3. Sometimes RAA arguments need to be carried out <u>within</u> another RAA argument.

Here is an example: ~(~P&~Q) $\vdash$ PvQ

Step 1. If you think about the problem a little, it can be seen that it will be impossible to prove either P or Q alone and then use vI. Our only hope is RAA. So I will assume ~(PvQ) and try to derive a contradiction.

1	(1) ~(~P&~Q)	A
2	(2) ~(PvQ)	A
	new goal is a contradiction	
	PvQ	RAA

Step 2. Which contradiction to try to prove? If we want to make it easy on ourselves, we can try to contradict something that we already have in the proof. Contradicting line 2 would mean proving PvQ which is exactly the problem that I am in the middle of. So I will try to contradict line 1 by proving ~P&~Q. This is a conjunction so I will try to prove each conjunct separately and then use &I.

1	(1) ~(~P&~Q)	A
2	(2) ~(PvQ)	A
	~P	
	~Q	
	~P&~Q	&I
	PvQ	RAA

Step 3. My new goals are ~P and ~Q. Let's think just about ~P for now. RAA seems the only reasonable strategy here. That means we should assume P and try to prove a contradiction.

1	(1) ~(~P&~Q)	A
2	(2) ~(PvQ)	A
3	(3) P	A
	new goal is a contradiction	

		~P	RAA
		~Q	goal
		(n-1) ~P&~Q	&I
		(n) PvQ	RAA

Step 4. The problem is now to derive a contradiction. If we decide to attempt to contradict line 2, it is, it is easy to see that this proof has transformed into a problem that is analogous to EXAMPLE 2. We just have P instead of ~P and Q instead of ~Q.	1	(1) ~(~P&~Q)	A
	2	(2) ~(PvQ)	A
	3	(3) P	A
	3	(4) PvQ	3 vI
	2	(5) ~P	2,4 RAA (3)
	6	(6) Q	A
	6	(7) PvQ	6 vI
	2	(8) ~Q	2,7 RAA (6)
	2	(9) ~P & ~Q	5,8 &I
	1	(10) PvQ	1, 9 RAA (2)